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NEW CLASSES OF SIMILAR SOLUTIONS FOR LAMINAR FREE CONVECTION PROBLEMS†

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Abstract—As shown by the senior author, the proper formulation of free convection boundary-layer theory depends on order of magnitude of the Eckert number defined as $Ec = Hg/c_p\Delta T$, the conventional theory being valid in the limit $Ec \to 0$. The present paper investigates the solutions of the laminar flat plate problem, over the entire Ec-range, for the case in which similarity prevails. It is shown that for $Ec \equiv O(1)$ the similar solutions are attainable for linearly varying wall temperature (in particular constant) whereas in the limit for $Ec \to \infty$ any wall temperature distribution leads to similar solutions. Similar profiles for $Ec \equiv O(1)$ depend on the Prandtl number and on the ratio (Ec/β') where β' is the constant wall temperature gradient. Similar profiles for $Ec \to \infty$ are universal insofar as they do not depend on any parameter. Universal profiles are given in closed form. Numerical solutions for $Pr = 0.72$ and several values of (Ec/β') are presented and analysed in terms of velocity and temperature profiles, wall shear stress and Nusselt number. In particular the paper shows that the results of conventional theory cannot be used for β' smaller than (0.05–0.1) Ec.

NOMENCLATURE

a_i, dimensionless constants;
C, arbitrary constant;
c_p, specific heat at constant pressure;
Ec, Eckert number ($= Hg/c_p\Delta T$);
Ec', differential Eckert number ($= g\,dx/c_p\,dT_w$);
e, f, g, dimensionless functions;
F, G, dimensionless functions;
g, acceleration due to gravity;
Gr, Grashof number ($= g\Delta TH^3/\nu^2 T_a$);
H, plate height;
l, l_u, l_y, scale factors;
M^2, dimensionless coefficient ($= Hg/RT_a$);
Nu, Nusselt number;
p, pressure;
Pr, Prandtl number;
R, gas constant;
Ra, Rayleigh number ($= GrPr$);
T, temperature;
ΔT, temperature difference [$= T_w(0) - T_a$];
u, v, velocity components;
$\bar{v}$, scale factor;
x, y, cartesian coordinates;
W, dimensionless function.

Greek symbols

α, thermal diffusivity coefficient;
$\beta(\xi)$, dimensionless function;
γ, specific heat ratio;
δ, dimensionless function;
ε, scale factor [$= (Ra)^{-1/4}$];
$\bar{\varepsilon}$, scale factor [$= (EcRa)^{-1/4}$];
η, similarity variable [$= y^*/\delta^*(\xi)$];
θ, dimensionless coefficient [$= \Delta T/T_a$];
λ, thermal conductivity coefficient;
μ, dynamic viscosity coefficient;
ν, kinematic viscosity coefficient;
ξ, similarity variable;
ρ, density;
σ, dimensionless function;
τ, shear stress;
ϕ, dimensionless function;
ψ, stream function;
ω, dimensionless constant.

Subscripts

a, ambient;
w, at the wall.

Superscripts

e, in the outer region;
$*$, dimensionless quantity;
$\bar{}$, dimensionless quantity for $Ec \gg 1$.

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1. INTRODUCTION

MATHEMATICAL modeling and solution of laminar free convection boundary layers are classical problems, recently reviewed by Ede [1]. The great majority of work is based on the Schmidt–Beckman [2] formulation (herein referred to as "conventional formulation") according to which the non-dimensional flow features depend only on the Prandtl (Pr) and Grashof (Gr) (or Rayleigh $Ra = GrPr$) numbers. The conventional formulation is rarely questioned even when higher boundary-layer theories are developed [3]. Its range of validity has never been properly identified and, perhaps as a natural consequence, not enough attention has been paid to the question of the appropriate modeling outside this range.

This question has been recently addressed to by the senior author [4] for the particular case of the steady laminar free-convection boundary layer due to a vertical impermeable flat plate immersed in a perfect gas. The study was based on a rigorous order of magnitude analysis and on the concomitant application of the matched asymptotic expansion technique. The results shed a completely new light on free convection boundary-layer problems, revealing the need for "revisiting" all of them and indicating the avenues along which further theoretical and experimental work should be done.

To put the present work in the proper perspective some of the most relevant findings of the analysis performed in [4] will be summarized.

Under the above mentioned restrictive hypotheses, the free convection problem exhibits five characteristic speeds: the Torricellian speed $V_g = (Hg)^{1/2}$, the speeds $V_\nu = \nu/H$ and $V_\alpha = \alpha/H$ related to diffusion of momentum and energy, and the two "thermal" speeds $V_a = (RT_a)^{1/2}$ and $V_T = (R\Delta T)^{1/2}$. The steady flow field thus depends, in general, upon four independent nondimensional parameters which can be formed with these five characteristic speeds. Since for gases $\nu/\alpha \equiv O(1)$ one such parameter, the Prandtl number $Pr = V_\nu/V_\alpha$, is of the order one and only the following other three parameters may have arbitrary orders of magnitude:

$$Ra = \frac{V_g^2}{V_a^2}\frac{V_T^2}{V_\nu V_\alpha} = \frac{g\Delta T H^3}{T_a \nu^2} Pr = Gr\,Pr$$

$$M^2 = \frac{V_g^2}{V_a^2} = \frac{Hg}{RT_a}; \quad \theta = \frac{V_T^2}{V_a^2} = \frac{\Delta T}{T_a}. \tag{1.1}$$

The first parameter is the classical Rayleigh number. The choice of the other two is actually motivated by their physical meanings. The parameter θ is a nondimensional measure of the "driving force" and affects directly the density field: for $\theta \ll 1$ the Boussinesq approximation is applicable. The parameter M^2 is the Newtonian Mach number referred to the Torricellian speed and thus affects directly the pressure field. When both $\theta \ll 1$ and $M^2 \ll 1$ the proper formulation of a boundary-layer theory still depends on the relative order of magnitude of θ and M^2 or, more precisely, on the order of magnitude of the Eckert number defined as

$$Ec = \frac{\gamma - 1}{\gamma}\frac{M^2}{\theta} = Hg/c_p\Delta T,$$

where γ is the specific heat ratio. The two key points to be stressed are:

(1) The conventional boundary-layer formulation is valid in the limit for $Ec \to 0$ and presupposes that $Ra \gg 1$. The same condition $Ra \gg 1$ applies for $Ec \equiv O(1)$. In the limit for $Ec \to \infty$ the validity of a boundary-layer theory depends no longer on the value of the Rayleigh number, but, rather, on that of the number:

$$EcRa = \frac{g^2H^4}{c_p T_a} Pr.$$

(2) The zeroth (and higher order) boundary-layer equations have different forms in each of the above three cases: $Ec \to 0$; $Ec \equiv O(1)$; $Ec \to \infty$. The implications of this state of affairs are of utmost importance as the following reasonings show.

Let all other problem data be constant and consider what happens as $\Delta T \to 0$. As this implies that $Ec \to \infty$, deductions inferred from conventional boundary-layer theory are erroneous. Thus, for example, it is not true that at any given station the boundary layer thickens indefinitely and that, consequently, the boundary-layer approach should necessarily fail to be applicable below a certain minimum value of ΔT.

The correct statements follow from point (1) above. When the number $EcRa$ is sufficiently large, a boundary-layer theory is applicable throughout the Ec range. Everything else being constant, as $\Delta T \to 0$ the boundary layer at a given station does not thicken indefinitely but reaches, asymptotically, a value proportional to $(EcRa)^{-1/4}$ which is constant since, appropriately, $EcRa$ is independent of ΔT.

Similar remarks apply for any property of the flow field such as mass flux in the boundary layer, wall shear stress, wall heat transfer and so on.

The essence of the above-mentioned two key points pertains (with few appropriate modifications, if any) to all free convection problems. Hence, the research avenues to be pursued appear clear. For each specific problem the other two sets of boundary-layer equations, for $Ec \equiv O(1)$ and $Ec \to \infty$ should be derived and solved. Solutions of the set of equations for $Ec \equiv O(1)$ provide, for any flow feature, the fairing between the two different functional expressions obtained in the two limits $Ec \to 0$ (as already available from the conventional theory) and $Ec \to \infty$. These solutions will yield quantitative information on the Ec-rate of variation of the field features. Thus ultimately they will provide a quantitative assessment on the exact range of validity of the conventional solution (which may well turn out to be different features of the field) as well as corrections to be applied for Ec small.

The present paper offers a first contribution in this direction by investigating the solutions of the laminar flat plate problem, over the entire Ec-range, for the case in which similarity prevails.

In paragraph (2) the different sets of zeroth order boundary-layer equations are briefly derived both for completeness sake and to further elucidate a number of essential points. Similarity forms are discussed in paragraph (3); solutions are presented and analysed in paragraph (4). The main conclusions are reviewed and summarized in the last paragraph.

2. BOUNDARY-LAYER EQUATIONS

Only the zeroth order approximations for the outer and inner regions will be considered. The coordinate systems has the origin at the lower edge of the vertical plate with the x-axis pointing upward. The gas is taken to be perfect and bulk viscosity effects are neglected.

The following non-dimensional (asterisked) quantities are introduced:

$$\begin{aligned} x = Hx^*;\ y = l_y H y^*;\ \rho = \rho_a(x=0)\rho^* \\ p = p_a(x=0)p^*;\ u = l_u V_a u^*;\ v = l_u l_y V_a v^* \\ T = \Delta T T^* + T_a;\ \Delta T = T(0,0) - T_a = T_w(0) - T_a \\ \mu = \mu_a \mu^*;\ \lambda = \lambda_a \lambda^*. \end{aligned} \tag{2.1}$$

The scales

$$V = l_u V_a;\ l = l_y H \tag{2.2}$$

for vertical component of the velocity and for the horizontal distance from the plate are left for the moment unspecified since their appropriate expression depends on the order of magnitude of the Eckert number. The scale for the horizontal component of the velocity is fixed by the requirement that horizontal and vertical convection be of the same order of magnitude.

The zeroth order outer solution (subscript e) describes a quiescent isothermal ambient and is given by:

$$\begin{aligned} u_e^* = v_e^* = T_e^* = 0 \\ p_e^* = \rho_e^* = \exp(-M^2 x^*). \end{aligned} \tag{2.3}$$

The equations for the zeroth order inner field can be written (upon neglecting terms of order l_y^2 or smaller and on accounting for the fact that, to within this approximation, $p^*(x^*, y^*) = p_e^*(x^*)$):

$$\begin{aligned} &(\rho^* u^*)_{x^*} + (\rho^* v^*)_{y^*} = 0 \\ &\rho^* T^* = \frac{l_u^2}{M^2\theta}\left[\rho^* \mathbf{V}^* \cdot \nabla^* u^* - \left(\frac{M^2\theta Pr}{Ra l_u^2 l_y^4}\right)^{1/2} (\mu^* u_{y^*}^*)_{y^*}\right] \\ &p_{y^*}^* \equiv O(l_u^2 l_y^2) \\ &(\lambda^* T_{y^*}^*)_{y^*} = \left(\frac{Ra Pr l_u^2 l_y^4}{\theta M^2}\right)^{1/2} \rho^* \mathbf{V}^* \cdot \nabla^* \left[T^* + \frac{(\gamma-1) l_u^2}{2\gamma\theta} u^{*2} + Ec x^*\right] - \frac{Pr l_u^2}{\theta}\frac{\gamma-1}{\gamma}(\mu^* u^* u_{y^*}^*)_{y^*} \\ &\rho^*(1+\theta T^*) = \rho_e^* = \exp(-M^2 x^*) \end{aligned} \tag{2.4}$$

where

$$\mathbf{V}^* \cdot \nabla^* = u^* \frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*}$$

and the numbers Ra, Pr and Ec are evaluated at the condition of the bulk fluid for $x = 0$.

The fourth equation expresses the conservation of total energy, sum of enthalpy, kinetic and potential energies. The latter is given by $\varphi = gx = V_g^2 x$ and is referred to its value at $x = 0$.

Equations (2.4) represent the zeroth order term of an asymptotic series expansion in terms of the small parameter $\varepsilon = l_y$. As this parameter is just the scale factor l_y, its explicit expression will depend on the order of magnitude of the Eckert number.

Further discussions will be restricted to the case in which $M^2 \ll 1$ and $\theta \ll 1$. These conditions imply the validity of the approximation $\rho^* = \mu^* = \lambda^* = 1$ so that the velocity field is solenoidal. The order of magnitude of the Eckert number determines how the pressure plus body force T^* and thermal diffusion $T_{y^*y^*}^*$ terms are balanced in the momentum and energy equations. The following three cases appear relevant:

$$(1)\ Ec \ll 1;\quad (2)\ Ec \equiv O(1);\quad (3)\ Ec \gg 1. \tag{2.5}$$

In the first two cases momentum and energy balance require that (unless inessential constants of order one):

$$\frac{l_u l_y^2}{M}\left(\frac{Ra Pr}{\theta}\right)^{1/2} = 1;\quad \frac{l_u^2 Pr}{M^2\theta} = 1.$$

These relations determine l_u and l_y and, consequently, the velocity and length scales and the expansion parameter which are given by:

$$\begin{aligned} V = l_u V_a = \left(\frac{Hg\Delta T\alpha}{\nu T_a}\right)^{1/2} = \frac{\alpha}{H}(Ra)^{1/2} \\ l = l_y H = H/(Ra)^{1/4};\quad \varepsilon = l_y = (Ra)^{-1/4}. \end{aligned} \tag{2.6}$$

The corresponding set of zeroth order boundary-layer equations is:

$$\begin{aligned} u_{x^*}^* + v_{y^*}^* = 0 \\ \frac{1}{Pr}\mathbf{V}^* \cdot \nabla^* u^* = u_{y^*y^*}^* + T^* \\ \mathbf{V}^* \cdot \nabla^* T^* + Ec u^* = T_{y^*y^*}^* \\ \rho^* = 1. \end{aligned} \tag{2.7}$$

The boundary conditions pertinent to an impermeable plate with non-uniform temperature are:

$$\begin{aligned} u^*(x^*, 0) = v^*(x^*, 0) = 0 \\ \lim_{y^* \to \infty} u^*(x^*, y^*) = \lim_{y^* \to \infty} T^*(x^*, y^*) = 0 \\ T^*(x^*, 0) = W^*(x^*);\quad [W^*(0) = 1]. \end{aligned} \tag{2.8}$$

The conventional formulation is valid only when $Ec \ll 1$ [first case in (2.5)] and is formally recovered from equations (2.7) by setting $Ec = 0$.

When $Ec \equiv O(1)$ [second case in (2.5)] the potential energy is of the same order as the enthalpy and its contribution to the energy conservation equation cannot be neglected. The term Ecu^* introduces a stronger coupling between velocity and temperature fields. It represents the rate of work done by the body force on the particle during its motion, equal to minus the time rate of change of its potential energy. The relevance of this term is inequivocally established when the total energy conservation equation is used. When, instead, one uses the balance equation for the internal energy, particular care must be exercised in handling the reversible work term $p\nabla \cdot \mathbf{V}$. It is speculated that one of the reasons why the system (2.7) has never been considered before is because the internal energy balance equation is usually employed and, for $\theta \ll 1$, the $p\nabla \cdot \mathbf{V}$ term is much too hurriedly eliminated on the ground that $\nabla \cdot \mathbf{V} = 0$. This inference is valid only for $Ec \ll 1$. The correct reasoning to be applied in the other cases is discussed in [4] and, obviously, leads to the same set of equations (2.7).

In the third case ($Ec \gg 1$) the potential energy is more relevant than both the enthalpy and the kinetic energy so that thermal diffusion can only be balanced by the rate of work done by the body force.

As equations (2.4) show, the appropriate balancing requires that:

$$\frac{Ecl_u l_y^2}{M}\left(\frac{RaPr}{\theta}\right)^{1/2} = 1;\quad \frac{l_u}{Ml_y^2}\left(\frac{Pr}{\theta Ra}\right)^{1/2} = 1 \quad (2.9)$$

from which the following new expressions $\bar{V}$, $\bar{l}$ and $\bar{\varepsilon}$ for the scales and the expansion parameter are obtained:

$$\bar{V} = \Delta T\left(\frac{\lambda}{\mu T_a}\right)^{1/2} = Ec^{-1/2}V$$

$$\bar{l} = \frac{H}{(EcRa)^{1/4}} = Ec^{-1/4}l;\quad \bar{\varepsilon} = (EcRa)^{-1/4} = Ec^{-1/4}\varepsilon.$$

The pertinent equations are:

$$\bar{u}^*_{\bar{x}^*} + \bar{v}^*_{\bar{y}^*} = 0$$
$$\bar{u}^*_{\bar{y}^*\bar{y}^*} + \bar{T}^* = 0 \quad (2.10)$$
$$\bar{T}^*_{\bar{y}^*\bar{y}^*} - \bar{u}^* = 0$$

where a bar denotes quantities referred to the new scales (2.9). Boundary conditions are as in equation (2.8) in terms of barred quantities. Equations for $Ec \gg 1$ were incompletely stated in [4] since only the particular case $\bar{v}^* = 0$ was reported. All convection terms have disappeared and gravitational effects are solely balanced by diffusion in both momentum and energy equation. The $\bar{v}^*$ field is uncoupled from the $\bar{u}^*$ and $\bar{T}^*$ fields.

As equations (2.9) show, the set of equations (2.10) can be obtained from the set (2.7) by performing the following changes in velocity and length scales:

$$\bar{y}^* = (Ec)^{1/4}y^*;\ \bar{u}^* = (Ec)^{1/2}u^*;\ \bar{v}^* = (Ec)^{3/4}v^* \quad (2.11)$$

and by neglecting terms of order $(Ec)^{-1}$ or smaller.

As mentioned in the introduction, the difference in the velocity and length scales for the two limits $Ec \to 0$ and $Ec \to \infty$ imply different functional behaviours of the flow properties, the fairing being afforded by the solutions of equations (2.7) for different values of Ec. Some further elaboration of this point may be appropriate.

Consider for instance the expressions (2.6) and (2.9) for the velocity scale which will be rewritten as:

$$V\left(\frac{Pr}{RT_a}\right)^{1/2} = \begin{cases} \left(\frac{Hg}{RT_a}\right)^{1/2}\left(\frac{\Delta T}{T_a}\right)^{1/2}: & Ec \leqslant O(1) \quad (2.12) \\ \left(\frac{\gamma}{\gamma-1}\right)^{1/2}\frac{\Delta T}{T_a}: & Ec \gg 1. \quad (2.13)\end{cases}$$

At a given station, the maximum values of the vertical component of the velocity referred to $V(Pr/RT_a)^{1/2}$ is proportional to the values given by equations (2.12) and (2.13) in the limits for $Ec \to 0$ and $Ec \to \infty$ respectively. Hence, everything else being constant, this maximum value of the upward velocity tends to be proportional to $(\Delta T/T_a)^{1/2}$ as $Ec \to 0$ (i.e. for sufficiently large values of $\Delta T/T_a$) as indeed predicted by the conventional theory, whereas it tends to be proportional to $\Delta T/T_a$ as $Ec \to \infty$ (thus, in particular, it goes to zero as $\Delta T/T_a$). How rapidly it goes from one functional dependence to the other can only be assessed by actually solving the system of equations (2.7).

Similarly, everything else being constant, the maximum value of the vertical component of the velocity, at a given station, tends to become proportional to $(Hg/RT_a)^{1/2}$ as Hg/RT_a tends to zero ($Ec \to 0$) whereas it tends to a constant value, independent of Hg/RT_a, as this parameter increases ($Ec \to \infty$).

Similar remarks can be made with respect to any other field property.

3. SIMILARITY FORMS OF SOLUTIONS

Let:

$$\xi = x^* + C;\quad \eta = y^*/\delta(x^*)$$
$$T^*(x^*, y^*) = \beta(\xi)g(\eta)$$
$$\psi^*(x^*, y^*) = e(\xi)f(\eta) \quad (3.1)$$
$$u^* = \psi^*_{y^*} = \frac{e}{\delta}f'$$
$$v^* = -\psi^*_{x^*} = \frac{e\eta}{\delta}\frac{d\delta}{d\xi}f' - \frac{de}{d\xi}f$$

where C is an arbitrary constant, ψ^* is the nondimensional stream function, η the similarity variable, primes denote differentiation with respect to η and $\beta(\xi)$, $e(\xi)$, $\delta(\xi)$ are scale factors.

The definitions (3.1) are appropriate for the analysis of system (2.7). For that of system (2.10) the same definitions (3.1) apply in terms of barred quantities (e.g. $\eta = \bar{y}^*/\bar{\delta}$; $\bar{u}^* = \bar{e}f'(\eta)/\bar{\delta}$ and so on) with:

$$\bar{\beta} = \beta;\quad \bar{\delta} = Ec^{1/4}\delta;\quad \bar{e} = Ec^{3/4}e. \quad (3.2)$$

Appropriate substitutions into equations (2.7) and (2.10) lead to the following sets of ordinary differential equations:

$$\begin{cases} f''' + \frac{1}{Pr}(a_1 ff'' - a_2 f'^2) + a_3 g = 0 \\ g'' + a_1 fg' - a_4 f'g - a_5 Ecf' = 0 \end{cases} \quad (3.3a)$$

$$\begin{cases} f''' + \bar{a}_3 g = 0 \\ g'' - \bar{a}_5 f' = 0. \end{cases} \quad (3.3b)$$

Similarity conditions are expressed by the constancy of the a_i's and $\bar{a}_i$'s respectively defined as:

$$\begin{cases} a_1 = \delta\frac{de}{d\xi};\ a_2 = \delta^2\frac{d}{d\xi}\left(\frac{e}{\delta}\right);\ a_3 = \frac{\delta^3\beta}{e} \\ a_4 = \frac{\delta e}{\beta}\frac{d\beta}{d\xi};\ a_5 = \frac{\delta e}{\beta} \end{cases} \quad (3.4a)$$

$$\bar{a}_3 = \frac{\bar{\delta}^3\bar{\beta}}{\bar{e}};\quad \bar{a}_5 = \frac{\bar{\delta}\bar{e}}{\bar{\beta}}. \quad (3.4b)$$

The similarity condition on a_5 is not required in the conventional formulation ($Ec = 0$) and equations (3.4a) reduce then to the set solved by Sparrow and Gregg [5]. Conversely, for $Ec \gg 1$ only two similarity conditions need to be imposed. For $Ec \equiv O(1)$ the additional condition on a_5 restricts the classes of similarity solutions to the linear or constant wall temperature distributions. If β' is the constant rate of change of wall temperature, the general solutions of equations (3.4a) are given, when suitably normalized, by

$$\beta(\xi) = 1+\beta'\xi;\ \delta = Ec^{-1/4};\ e(\xi) = \frac{1+\beta'\xi}{Ec^{3/4}} = \frac{\beta}{Ec^{3/4}} \quad (3.5)$$

and the corresponding form of equations (3.3a) reads:

$$\begin{cases} f''' + \dfrac{\beta'}{Pr\,Ec}(ff''-f'^2)+g = 0 \\ g'' + \dfrac{\beta'}{Ec}(fg'-f'g)-f' = 0. \end{cases} \quad (3.6)$$

For $Ec \gg 1$ the reduced number of similarity conditions to be imposed enlarges the classes of similarity solutions to include any distribution of wall temperature. Indeed the general (normalized) solution of equations (3.4b) is simply:

$$\bar{\delta} = 1;\quad \bar{\beta}(\xi) = \bar{e}(\xi). \quad (3.7)$$

The corresponding equations read:

$$\begin{cases} f'''+g = 0 \\ g''-f' = 0. \end{cases} \quad (3.8)$$

In both cases the boundary conditions are:

$$\begin{cases} f(0) = f'(0) = 0;\quad g(0) = 1 \\ \lim\limits_{\eta\to\infty} f' = \lim\limits_{\eta\to\infty} g = 0. \end{cases} \quad (3.9)$$

Equations (3.6) can be given in several alternative forms by subjecting them to appropriate changes in the dependent and independent variables. All of them are equivalent when $Ec \equiv O(1)$ but they would correspond to sets of velocity and length scales of different orders of magnitude when Ec is not of order one. Substitution of equations (3.5) into equation (3.1) shows that, for $Ec \gg 1$, the scales corresponding to equations (3.6) are just the scales $\bar{l}$ and $\bar{V}$ appropriate to the limiting case $Ec \to \infty$. This is further evidenced by the fact that equations (3.8), holding for $Ec \gg 1$, can be formally obtained from equations (3.6) by performing on them the limit for $Ec \to \infty$.

Among all alternate forms of equations (3.6) a relevant role is played by the one which corresponds to the scales l and V appropriate to the other limit for $Ec \to 0$. Such a form obtained, as clearly suggested by equations (2.11), when the following changes are performed:

$$\begin{aligned} f(\eta) &= \left(\frac{Ec}{\beta'}\right)^{3/4} F(\sigma) \\ g(\eta) &= G(\sigma) \\ \eta &= \left(\frac{Ec}{\beta'}\right)^{1/4} \sigma. \end{aligned} \quad (3.10)$$

The new form of equations (3.6) reads:

$$\begin{cases} F''' + \dfrac{1}{Pr}(FF''-F'^2)+G = 0 \\ G''+FG'-F'G-\dfrac{Ec}{\beta'}F' = 0 \end{cases} \quad (3.11)$$

and they are subject to the same type of boundary conditions given by equations (3.9). As a check, notice that equations (3.11) reduce, in the limit for $Ec \to 0$, to the conventional equations holding for linearly varying wall temperature. Equations (3.6) and (3.11) correspond to two different representations of the field variables which, together, cover the entire range of Ec from 0 to ∞. These representations are obtained by combining equations (3.5) with equations (3.1), (2.6) and (2.1) for one representation and by further performing the transformation (3.10) for the other one. The result is:

$$\begin{aligned} u(x,y) &= \frac{\alpha}{H}\left(\frac{Ra}{\beta'}\right)^{1/2} \beta F'\left(\sigma, Pr, \frac{Ec}{\beta'}\right) \\ &= \frac{\alpha}{H}\left(\frac{Ra}{Ec}\right)^{1/2} \beta f'\left(\eta, Pr, \frac{\beta'}{Ec}\right) \\ v(x,y) &= -\frac{\alpha}{H}\left(\frac{Ra}{\beta'^3}\right)^{1/4} \beta' F\left(\sigma, Pr, \frac{Ec}{\beta'}\right) \\ &= -\frac{\alpha}{H}\left(\frac{Ra}{Ec^3}\right)^{1/4} \beta' f\left(\eta, Pr, \frac{\beta'}{Ec}\right) \\ \frac{T-T_a}{T_w(0)-T_a} &= \beta(\xi) G\left(\sigma, Pr, \frac{Ec}{\beta'}\right) \\ &= \beta(\xi) g\left(\eta, Pr, \frac{\beta'}{Ec}\right) \\ x &= Hx^* = H(\xi - G) \\ y &= \frac{H\sigma}{(\beta' Ra)^{1/4}} = \frac{H\eta}{(Ec\,Ra)^{1/4}} \end{aligned} \quad (3.12)$$

where the dependence of the similarity functions on the parameter appearing in the corresponding equation is explicited out.

The more relevant features of the subject–similar flow fields can now be discussed. As for the general case, what matters is the relative importance, in the energy conservation equation, of the contributions due to convection and time rate of work done by the body force.

The point of qualifying the similar fields is that the latter contribution is proportional to Ec and the former to β' (as a consequence of the constraints imposed by similarity conditions).

Hence, as equations (3.12) show, Ec and β' play a "symmetrical role" in the definitions of the scale factors whereas the similarity profiles depend upon Ec and β' only through their ratio $Ec' = Ec/\beta'$.

The different scale factors in the two representations of the independent and dependent variables are obtained by interchanging Ec and β' wherever they appear combined with the Rayleigh number Ra (which is the key scale parameter).

By their very nature, the similarity functions F, G, f, g cannot depend on overall quantities such as H and ΔT. This is appropriately reflected by their dependence on the ratio Ec'. This ratio can indeed be interpreted as a "differential" Eckert number for, when the appropriate substitutions are made, one gets:

$$Ec' = Ec/\beta' = \frac{g\,dx}{c_p\,dT_w}. \quad (3.13)$$

The different roles played by Ec and β' in the scale factors and in the similarity functions are fundamental in analysing the limits for $Ec' \to 0$ and $Ec' \to \infty$. For β' finite and $Ec \to 0$ the first representation in equation (3.12) is appropriate. In this limit β' disappears from the differential equations (3.11) which reduce to those solved by Sparrow and Gregg [5]. One thus recovers the results of conventional theory for a linear distribution of wall temperature since the contribution due to body force power can be neglected in the energy conservation equation. The wall temperature gradient β' enters only in the definition of the scales and is the "controlling" factor in determining the limits of applicability of a boundary-layer theory. For the subject similar flow field, these limits depend on the value of the parameter:

$$\beta' Ra = \frac{gH^4}{T_a \nu^2} \frac{dT_w}{dx} Pr \qquad (3.14)$$

which as Ec' can be interpreted as a "differential" form of the Rayleigh number. Conversely, for $\beta' \to 0$ and Ec finite, $Ec' \to \infty$ and the contributions due to the convection terms can be neglected in both the momentum and energy equations. The representation to be used in investigating this limit is the second one. The factor "controlling" the different scales is Ec. In particular: the thickness of the boundary layer depends on the value of $EcRa$ which, as already pointed out, is independent of ΔT. For $Ec' \to \infty$ equations (3.6) reduce to equations (3.8): the similarity profiles are unique, their dependence on Pr, β', Ec having disappeared. Thus, in particular, the similarity profiles for a constant wall temperature $\beta' = 0$ and *any* non-nullified value of Ec are identical to the similarity profiles prevailing for *any* wall temperature distribution in the limit for $Ec \to \infty$. The differences are felt only in the length, velocity and temperature scales as given by the second representation in equations (3.12) (notice, in particular, that for a constant wall temperature the v-component scale vanishes identically).

The last relevant case to be discussed is when both $\beta' \to 0$ and $Ec \to 0$ with $Ec' = Ec/\beta'$ finite. Either one of the representations (3.12) is appropriate: the two sets of scales are essentially of the same order. The similar profiles depend on the particular (finite) value of Ec' and, consequently, in this range the conventional theory gives erroneous results. On the light of the remarks previously made on the interpretation of β' and Ec the situation appears to be exactly the same as that discussed, in the preceding paragraph, for general non-similar flow fields.

The results of the conventional theory, as given by the functions $F(\sigma, Pr, 0)$, $G(\sigma, Pr, 0)$, are valid only for $\beta' \gg Ec$ and thus cannot be extrapolated down to $\beta' = 0$. No matter how small Ec is, as β' decreases one must eventually switch from the first to the second representation. This for instance implies that the boundary layer does not thicken indefinitely but reaches asymptotically a value proportional to $H/(EcRa)^{1/4}$ (independent of β'), and that the scale for the u-component of the velocity remains finite as $\beta' \to 0$. In other and more general words, as $\beta' \to 0$, Ec becomes the controlling factor for the scales and the similar profiles tend toward the "universal" profiles defined by equations (3.8).

Two closely related questions need further investigations and will be only briefly mentioned here. The first one concerns the commutativity of the operations (i) limit for $Ec \to 0$ and (ii) imposition of similarity conditions, which are performed in the above order in the conventional theory and in the reversed order in the present approach. The two operations commute only in the case of linear wall temperature distribution albeit, as seen, the β'-range of applicability of the results of conventional theory is bounded from below.

The present investigation suggests that an analogous limitation holds for the other classes of similar solutions given by the conventional theory. There must be a parameter which measures the relative importance of convection terms and results of conventional theory are applicable only when this parameter is much larger than Ec. Outside this range, the conventional formulation is no longer valid and, in addition, the flow field is no longer similar. To investigate these cases f and g in equations (3.1) must be considered also functions of ξ and the appropriate analysis is to be performed on the partial differential equations resulting from such substitution into the original system (2.4).

The constant wall temperature needs special mention since it is a somewhat "singular" case. Indeed, strictly speaking, the conventional similar solution applies only when both dT_w/dx and Ec are exactly zero. A totally different similar solution (namely the one exhibiting the universal profiles) is obtained for dT_w/dx exactly zero when Ec is different from zero, no matter how small. This solution is valid only for $EcRa$ sufficiently large: hence, for Ec small, it will prevail, if at all, in a region far away from the leading edge. Thus the problem to be investigated quantitatively is the entity and nature of the non-similarity corrections which must be performed on the conventional constant wall temperature similarity solution for $Ec \ll 1$ but not null.

The second question refers to the linear wall temperature distribution and concerns the sign of β'. The first representation is applicable as previously given only for $\beta' > 0$. For $\beta' < 0$ one must replace β' with $(-\beta')$ in equations (3.10) and change the sign of the convection terms in equations (3.11). Similar solutions for $\beta' < 0$ pose, however, some problems with regard to the range of values of β' (if any) for which solutions satisfying the prescribed boundary conditions exist. Similar solutions of the conventional boundary-layer equations ($Ec = 0$) have been given only for $\beta' > 0$ but no explicit proof seems to have been published concerning their non-existence for $\beta' < 0$. On the other hand, preliminary investigations by the present authors have shown that solutions do exist for $\beta' < 0$ and Ec finite. The matter clearly needs further studies.

4. SIMILAR PROFILES

4.1. *Universal profiles*

The solution of system (3.7) with the boundary conditions (3.8) can be given in closed form and reads:

$$
\begin{aligned}
f(\eta, Pr, Ec' = \infty) &= -\frac{1}{\sqrt{2}}\left\{\exp\left(-\frac{\eta}{\sqrt{2}}\right)\left[\sin\frac{\eta}{\sqrt{2}}+\cos\frac{\eta}{\sqrt{2}}\right]-1\right\} \\
g(\eta, Pr, Ec' = \infty) &= \exp\left(-\frac{\eta}{\sqrt{2}}\right)\cos\frac{\eta}{\sqrt{2}}.
\end{aligned}
\tag{4.1}
$$

Substitution in the second representation (3.12) for the field variables leads to the following expressions for the wall shear stress τ_w and the local Nusselt number Nu:

$$
\begin{aligned}
\tau_w &= -\frac{\mu\alpha\beta}{(\sqrt{2})H^2}\frac{Ra^{3/4}}{Ec^{1/4}} \\
Nu &= -\frac{x}{\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{\beta x}{(\sqrt{2})H}(EcRa)^{1/4}.
\end{aligned}
\tag{4.2}
$$

portional to $(-\beta')$. Mass exchange between the boundary layer and the bulk fluid depends on the sign of the rate of change of wall temperature. For constant wall temperature ($\beta' = 0$) the boundary layer is isolated and the total upward mass flux in it is constant. In the region where $\beta' > 0$ (increasing wall temperature) mass is being entrained in the boundary layer. The converse occurs in regions where the wall temperature decreases: mass flows from the boundary layer into the bulk fluid. This behaviour of the subject similarity flow fields is readily explained when noticing that the total upward mass flux through any section is, as a simple calculation shows, proportional to the local temperature difference $T_w(x) - T_a$. Hence, when $T_w(x) - T_a$ increases with x, mass must be entrained in the boundary layer from the outer fluid, and vice versa.

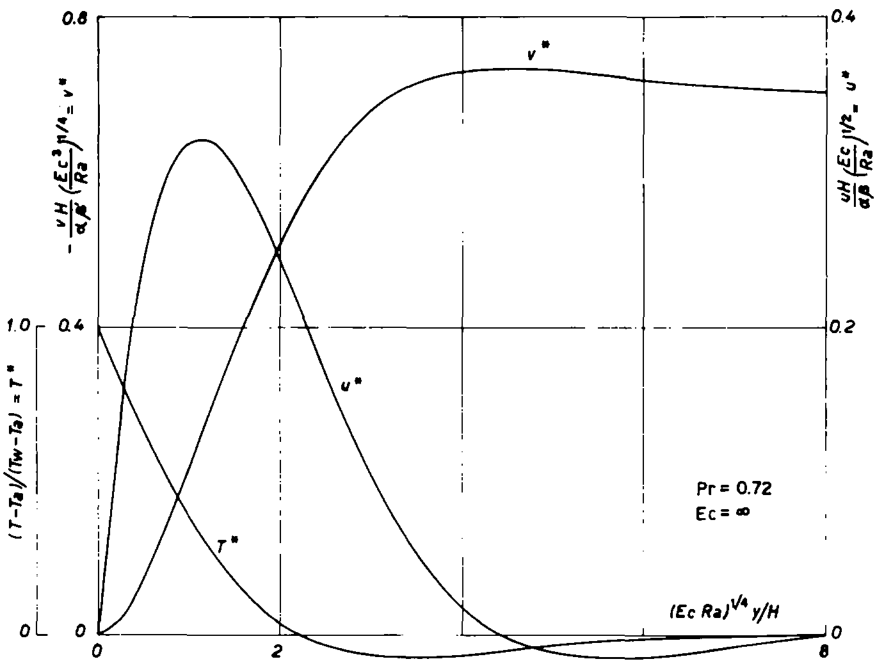

FIG. 1. Universal velocity components and temperature profiles for $Ec = \infty$.

The universal velocity and temperature profiles are plotted in Fig. 1 against $\eta/\sqrt{2}$.

In the upper two thirds of the (constant) boundary-layer thickness, the temperature is less than the bulk temperature. This is so because, as repeatedly mentioned, for $Ec' \gg 1$, convection effects are negligible, and pressure plus body force terms in the momentum equation are balanced only by the resultant of the viscous stresses. Hence, when the latter changes sign (namely at the inflection point of the u-profile) so does the temperature difference $(T - T_a)$. For exactly the same reason near the boundary-layer outer edge there is a region (above the inflection point of the temperature profile) where there is a small downward velocity. The v-component of the velocity at the outer edge is pro-

4.2. *Linear wall temperature distribution*

Equations (3.6) have been solved numerically with the automatic initial-value technique developed by Nachtsheims and Swigert [6] for several values of the Prandtl number Pr and of the parameter $Ec' = Ec/\beta' > 0$.

Results will be presented and discussed only for $Pr = 0.72$ since the Prandtl number, within its range of values appropriate to gases, does not affect appreciably the main features.

Velocity and temperature profiles for $Pr = 0.72$ are shown in Figs. 2–7. To exhibit more vividly the dependence of the similar profiles on the parameter Ec', the representation in terms of F, F', G has been used for $0 \leqslant Ec' \leqslant 1$ (Figs. 2–4), and that in terms of

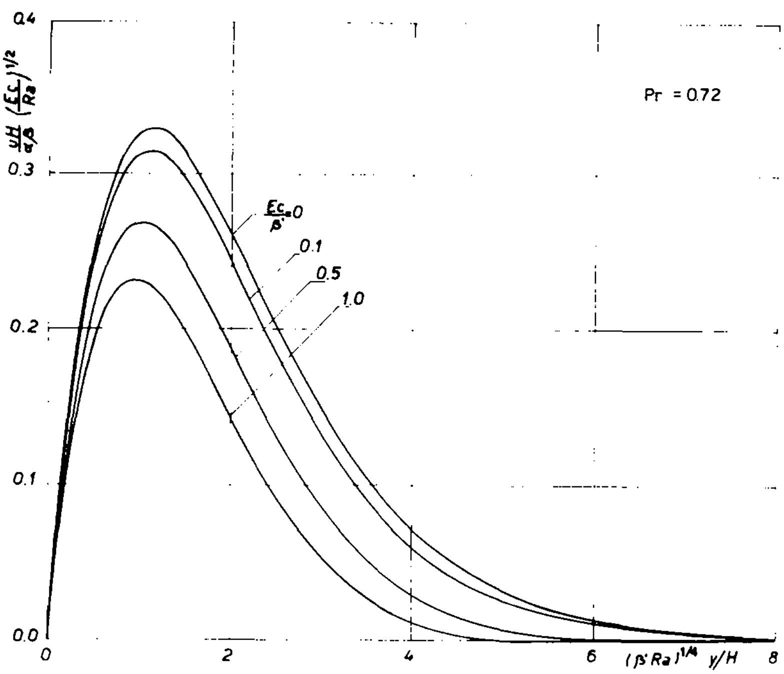

FIG. 2. Vertical velocity component profiles for $Pr = 0.72$, $Ec/\beta' \leqslant 1$ and linearly varying wall temperature ($Ec/\beta' = 0$ corresponds to conventional formulation).

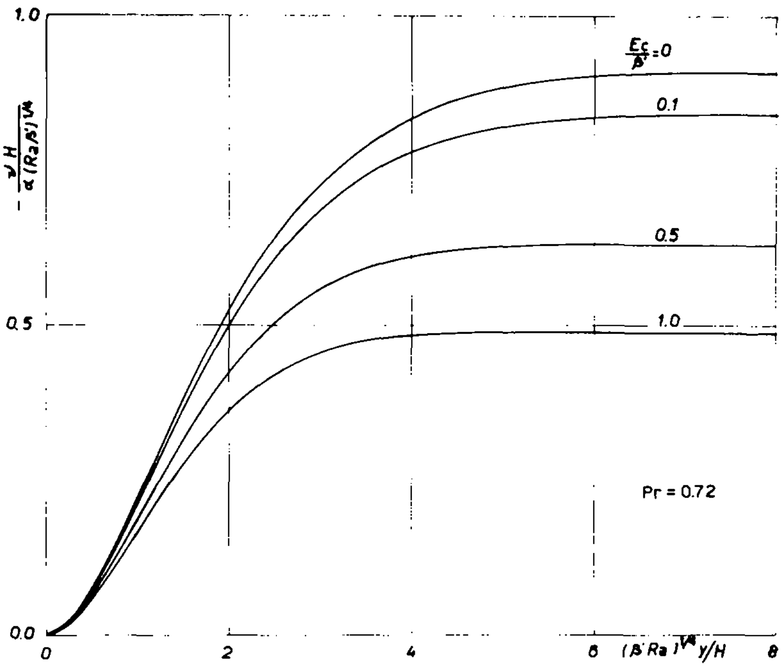

FIG. 3. Horizontal velocity component profiles for $Pr = 0.72$, $Ec/\beta' \leqslant 1$ and linearly varying wall temperature ($Ec/\beta' = 0$ corresponds to conventional formulation).

f, f', and g has been used for $1 \leqslant Ec' \leqslant \infty$ (Figs. 5–7).

Comparison between the two sets of figures is made easier if equations (3.12) are rewritten as:

$$U(x, y) = \frac{H}{\alpha\beta}\left(\frac{\beta'}{Ra}\right)^{1/2} u = F'(\sigma, Pr, Ec') = (Ec')^{-1/2} f'(\eta, Pr, Ec')$$

$$-V(x, y) = \frac{H}{\alpha}\frac{v}{(Ra\beta')^{1/4}} = F(\sigma, Pr, Ec') = (Ec')^{-3/4} f(\eta, Pr, Ec') \qquad (4.3)$$

$$\frac{T - T_a}{\beta[T_w(0) - T_a]} = \frac{T - T_a}{T_w(x) - T_a} = G(\sigma, Pr, Ec') = g(\eta, Pr, Ec')$$

$$\sigma = (Ec')^{-1/4}\eta.$$

Notice, in particular, that both G and g give the temperature difference in terms of the "local" wall tem-

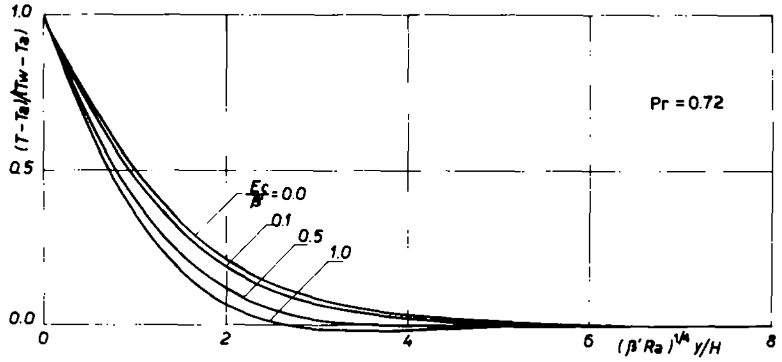

FIG. 4. Temperature profiles for $Pr = 0.72$, $Ec/\beta' \leqslant 1$ and linearly varying wall temperature ($Ec/\beta' = 0$ corresponds to conventional formulation).

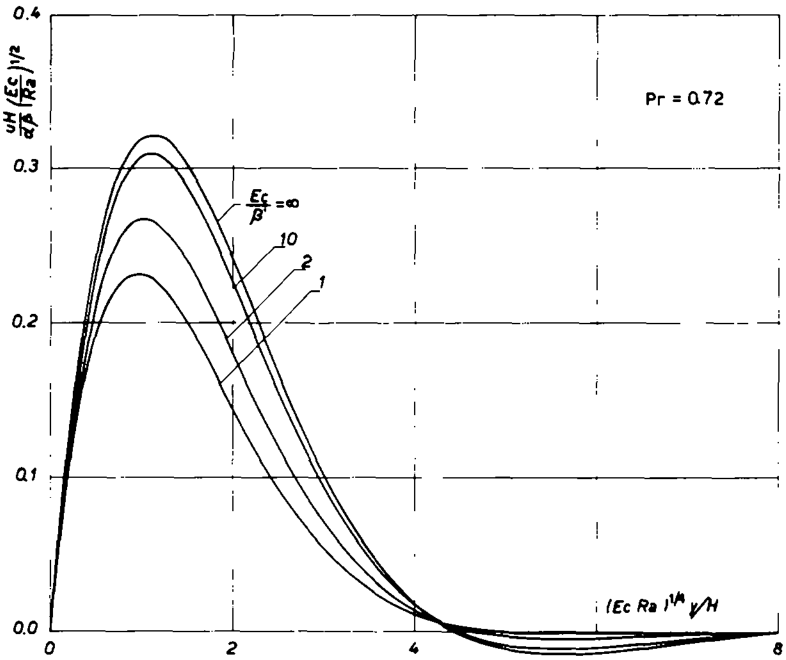

FIG. 5. Vertical velocity component profiles for $Pr = 0.72$, $Ec/\beta' \geqslant 1$ and linearly varying wall temperature ($Ec/\beta' = \infty$ corresponds to universal profile).

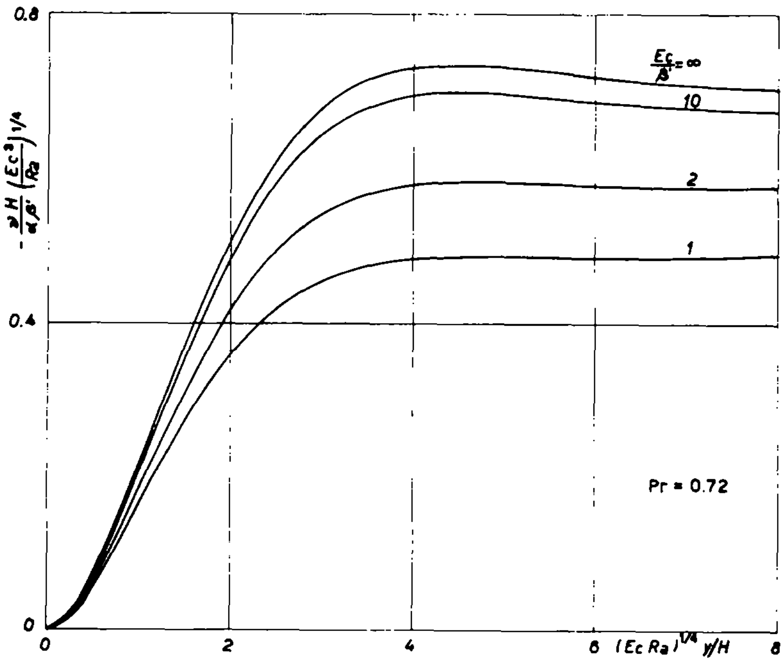

FIG. 6. Horizontal velocity component profiles for $Pr = 0.72$, $Ec/\beta' \geqslant 1$ and linearly varying wall temperature ($Ec/\beta' = \infty$ corresponds to universal profile).

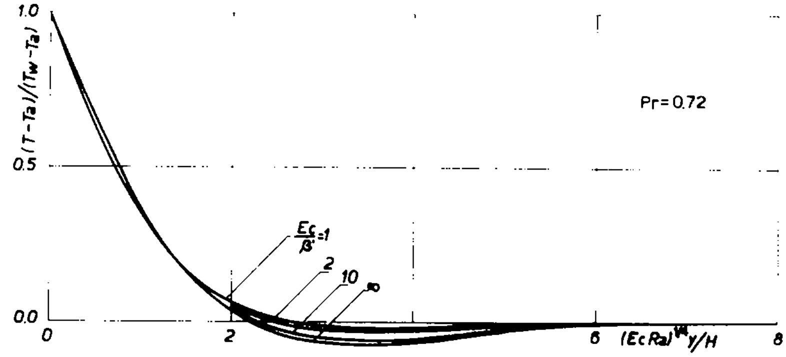

FIG. 7. Temperature profiles for $Pr = 0.72$, $Ec/\beta' \geq 1$ and linearly varying wall temperature ($Ec/\beta' = \infty$ corresponds to universal profile).

perature difference $T_w(x) - T_a$ and that the two sets of profiles coincide for $\bar{E}c' = 1$.

The profiles for $Ec' = 0$ are the "conventional" profiles given by Sparrow and Gregg. The effects of Ec' in the range (0, 1) are seen to be quite sizeable and comparatively of the same order for both velocity and temperature similar profiles. As Ec' increases the point where the u-component obtains a maximum moves closer to the wall and, quite expectedly, this maximum value decreases. For $Ec' = 1$ it is 70% of the conventional value. Similarly, $F(\infty)$ decreases as Ec' increases and for $Ec' = 1$ it is equal to 53% of the conventional value.

As Ec' increases, the values of the temperature similarity profile decreases and, consequently, the temperature normal gradient at the wall increases in absolute value. This is expected since, as Ec' increases, the rate of change of a particle's potential energy becomes increasingly more important and thus the energy transferred from the plate to the boundary layer must be spent not only to raise the particle's temperature but also its potential energy. The increase in energy transmitted by the plate will eventually become insufficient to perform both actions throughout the boundary layer so that, as Ec' increases, a percentage of the outer boundary will eventually have a temperature smaller than the bulk temperature. This outer region is already measurable for $Ec' = 1$.

Figures 5–7 show that the influence of Ec' on the functions f, f', g is comparatively smaller. This implies that most of its influence on u, v, T is accounted for by the factorization represented in equations (3.12). Near the plate: the temperature field is substantially independent of Ec' for $Ec' > 1$; Fig. 7 shows that the g-profiles for $1 \leq Ec' \leq \infty$ are practically indistinguishable in that region.

The influence of Ec' on the wall shear stress and on the Nusselt number is shown in Figs. 8 and 9 where the quantities $\bar{\tau}_w$ and $\bar{N}u$, defined by:

$$\bar{\tau}_w = \frac{H^2}{\mu\alpha\beta}\left(\frac{\beta'}{Ra^3}\right)^{1/4}\tau_w = F''(0, Pr, Ec')$$

$$= (Ec')^{-1/4} f''(0, Pr, Ec')$$

$$\bar{N}u = \frac{H}{\beta x}(\beta' Ra)^{-1/4} Nu = -G'(0, Pr, Ec')$$

$$= -Ec' g'(0, Pr, Ec')$$

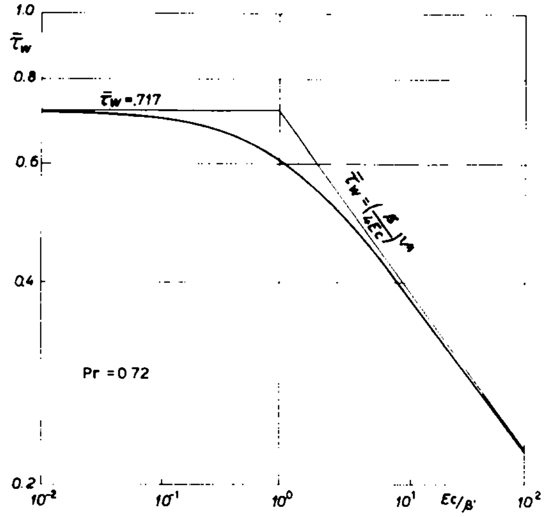

FIG. 8. Dimensionless shear stress at the wall vs the differential Eckert number Ec/β' for $Pr = 0.72$.

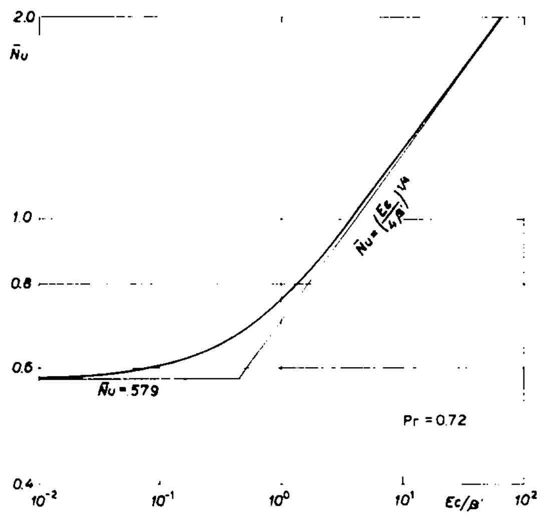

FIG. 9. Local Nusselt number vs the differential Eckert number Ec/β' for $Pr = 0.72$.

are plotted against Ec'. The two limiting behaviours are also shown. For $Ec' \to 0$, $\bar{\tau}_w$ and $\bar{N}u$ tend to values depending only on the Prandtl number. This dependence is too weak to be shown on the diagrams. For $Ec' \to \infty$, $\bar{\tau}_w$ and Nu tend to be proportional to $(Ec')^{-1/4}$ and $(Ec')^{1/4}$ respectively. The rate of deviation from the limiting behaviours is larger for $Ec' \to 0$ than for $Ec' \to \infty$ and this difference is more marked for $\bar{N}u$. Indeed, as Figs. 8 and 9 show, for $Ec' = 1$ the two limiting solutions (for $Ec' \to 0$ and $Ec' \to \infty$) yield practically the same value for $\bar{\tau}_w$, whereas for $\bar{N}u$, the value given by the limiting solution for $Ec' \to \infty$ is much closer to the exact value than that given by the other limiting solution. In other words, for low values of Ec', $\bar{N}u$ is more sensitive than τ_w to changes in Ec'.

All quantitative results presented in this paragraph show that the results given by the conventional theory for linearly varying wall temperature can be practically accepted up to values of Ec' between 0.05 and 0.1 [i.e. for $\beta' \geq (0.05\text{–}0.1)Ec$]. The errors thus made vary according to the field property being considered. For $Ec' = 0.1$ the maximum value of U is overestimated by 4.7%, $\bar{\tau}_w$ is overestimated by 2.4% and $\bar{N}u$ is underestimated by 4.3%.

5. CONCLUDING REMARKS

Similar solutions for the laminar flat plate free convection problems throughout the entire Ec-range have been investigated. Some of the most important conclusions are now reviewed and summarized.

The Eckert number measures the relative importance between potential energy and enthalpy of a particle (static interpretation) or between the power associated with body force and enthalpy convection (dynamic interpretation) and plays a relevant role in the characterization of similar flow fields.

When Ec is sufficiently large for the contribution of convection terms to be negligible (i.e. in the limit for $Ec \to \infty$), the number of similarity constraints to be imposed is minimum and any wall temperature distribution leads to similar fields. The similarity variable is independent of x; the similarity profiles are "universal" (i.e. do not depend on any parameter) and are given in closed form. These similar fields represent the x-wise asymptotic solutions to which any flat plate problem will eventually tend, provided: Ec is sufficiently large for convective terms to be negligible, $EcRa$ is sufficiently large for the boundary-layer approximation to be applicable and the flow remains laminar. When Ec is sufficiently small for the contribution of potential energy to be negligible (i.e. in limit for $Ec \to 0$) the zeroth order boundary-layer equations are those given by the conventional theory. The number of similarity constraints is intermediate and, as well known, similar solutions prevail for power law, and exponential distributions of wall temperature.

When non *a priori* assumption is made on the order of magnitude of Ec similarity constraints are to be imposed on all terms contributing to momentum balance and total energy-conservation and thus the class of similar solutions result the most limited. Similarity flow fields are obtainable only when the wall temperature varies linearly (or, in particular, is constant). The similarity variable is independent of the x-coordinate; the similarity functions depend on the Prandtl number and on the ratio $E' = Ec/\beta'$ where β' is the constant wall temperature gradient. Velocity and length scales are controlled by β' (resp. Ec) in the limit for $Ec \to 0$ (resp. $\beta' \to 0$) and $Ec' \to 0$ (resp. $Ec' \to \infty$). For $\beta' = 0$ and any non-vanishing value of Ec the similarity profiles coincide with the above-mentioned universal profiles. For $Ec' = 0$, $\beta' \neq 0$ the conventional similarity solution is recovered.

The linear wall temperature distribution with positive gradient is the only distribution for which the flow field is similar throughout the Ec range. The range of validity of the conventional solution for linear wall temperature is bounded from below in terms of the parameter $Ec' = Ec/\beta'$. Numerical solutions of the more accurate set of equations show that the practical lower limit for Ec' is between 0.05 and 0.1 depending on the field property being considered. This implies that conventional solutions cannot be used for values of the non-dimensional temperature gradient β' smaller than (0.05–0.1) Ec.

In all other cases flow fields are not similar throughout the Ec range. Results of the present analysis would indicate that all other conventional similarity solutions should have their range of validity bounded from below in a manner analogous to that found for the linear wall temperature.

This aspect will be treated extensively elsewhere since the corrections to be made to the conventional theory are no longer in similar form. The most striking example is given by the constant wall temperature case for which the flow field is strikingly similar only when Ec is exactly zero.

Subject of future investigation will also be the case of linearly decreasing wall temperature ($\beta' < 0$). Results of a preliminary analysis have shown that, contrary to what seems to be the case for the conventional theory, similar solutions do exist for $\beta' < 0$ and Ec finite.

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NOUVELLES CLASSES DE SOLUTIONS EN SIMILITUDE POUR LES PROBLEMES DE CONVECTION NATURELLE LAMINAIRE

Résumé—Comme l'a montré le premier auteur, la formulation exacte de la théorie de la couche limite en convection naturelle dépend de l'ordre de grandeur du nombre d'Eckert défini par $Ec = Hg/C_p\Delta T$, la théorie conventionnelle étant valable à la limite $Ec \to 0$. Le présent article examine les solutions du problème de la plaque plane laminaire, sur tout le domaine de variation de Ec, dans le cas où existe une similitude.

On montre que pour $Ec \equiv O(1)$ les solutions en similitude sont obtenues pour une température de paroi variant linéairement (en particulier pour une température constante) tandis qu'à la limite $Ec \to \infty$ toute distribution de température pariétale conduit à des solutions en similitude.

Les profils en similitude pour $Ec \equiv O(1)$ dépendent du nombre de Prandtl et du rapport (Ec/β') où β' est le gradient constant de température à la paroi. Les profils en similitude pour $Ec \to \infty$ sont universels en ce sens qu'il ne dépendent d'aucun paramètre. Les profils universels sont complètement déterminés.

Des solutions numériques pour $Pr = 0,72$ et plusieurs valeurs de (Ec/β') sont présentées et analysées en termes de profils de vitesse et de température, de tension de cisaillement pariétal et du nombre de Nusselt. L'article montre en particulier que les résultats de la théorie classique ne peuvent être utilisés pour β' inférieur à $(0,05 \div 0,1)$ Ec.

NEUE GRUPPEN VON ÄHNLICHKEITSLÖSUNGEN FÜR PROBLEME DER LAMINAREN, FREIEN KONVEKTION

Zusammenfassung—Wie von den Autoren früher gezeigt worden ist, hängt die geeignete Formulierung der Grenzschichttheorie für die freie Konvektion von der Größenordnung der Eckert-Zahl $Ec = Hg/c_p\Delta T$ ab, wobei die konventionelle Theorie für den Grenzfall $Ec \to 0$ gültig ist. Die vorliegende Arbeit untersucht unter Voraussetzung der Ähnlichkeit die Lösungen des laminaren Problems der ebenen Platte über den gesamten Bereich der Ec-Zahlen.

Es wird gezeigt, daß für $Ec = 0(1)$ Ähnlichkeitslösungen für linear veränderliche Wandtemperaturen (im speziellen konstante Wandtemperaturen) möglich sind, während, für $Ec \to \infty$ jede Wandtemperaturverteilung zu Ähnlichkeitslösungen führt.

Ähnliche Profile für $Ec = 0(1)$ hängen von der Prandtl-Zahl und dem Verhältnis (Ec/β') ab, wobei β' der konstante Gradient der Wandtemperatur ist. Ähnliche Profile für $Ec \to \infty$ sind insofern universell, als sie nicht von anderen Parametern abhängen. Universelle Profile werden in geschlossener Form angegeben.

Für $Pr = 0,72$ und mehrere Werte von (Ec/β') werden numerische Lösungen angegeben und anhand von Geschwindigkeits- und Temperaturprofilen, von Wandschubspannungen und Nusselt-Zahlen analysiert. Insbesondere wird gezeigt, daß die Ergebnisse der konvektionellen Theorie für β'-Werte kleiner als $(0,05$ bis $0,1)$. Ec nicht verwendet werden können.

НОВЫЕ КЛАССЫ АВТОМОДЕЛЬНЫХ РЕШЕНИЙ ЗАДАЧ ЛАМИНАРНОЙ СВОБОДНОЙ КОНВЕКЦИИ

Аннотация — Как показано первым автором, правильная формулировка теории пограничного слоя при свободной конвекции зависит от порядка величины числа Эккерта, определяемого как $Ec = Hg/c_p\,\Delta T$, причем общепринятая теория является справедливой в пределе $Ec \to 0$. В настоящей статье рассматриваются преимущественно автомодельные решения задачи о ламинарном обтекании плоской пластины для широкого диапазона изменения числа Ec. Показано, что при $Ec = 0(1)$ автомодельные решения можно получить для линейно изменяющейся температуры стенки (в частности, постоянной), в то время как при $Ec \to \infty$ любое распределение температуры стенки приводит к автомодельным решениям. Автомодельные профили при $Ec = 0(1)$ зависят от значения числа Прандтля и отношения (Ec/β'), где β' — постоянный градиент температуры стенки. Автомодельные профили при $Ec \to \infty$ универсальны, так как не зависят ни от одного из параметров. Универсальные профили даны в замкнутом виде. Представлены численные решения для случая $Pr = 0,72$ и нескольких значений (Ec/β'), а также их анализ с помощью профилей скорости и температуры, касательного напряжения на стенке и числа Нуссельта. В статье, в частности, показано, что классическая теория неприемлема при значениях β' ниже $(0,05 \div 0,1)Ec$.